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Strong LP duality in weighted infinite bipartite graphs

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Abstract

We prove a weighted generalization of König's duality theorem for infinite bipartite graphs and a weighted version of its dual.

1. Introduction

Suppose that $\Gamma=(V,E)$ is a finite bipartite graph with a nonnegative integral weight function w on its edge set. Let A be the incidence matrix of Γ, i.e., A is a 0, 1 matrix on $V\times E$ such that $a_{ve}=1$ if $v\in e$, $e\in E$, $a_{ve}=0$ otherwise. A nonnegative real function x on E is called a *fractional matching* if $\sum_{v\in e}x(e)\leqslant 1$ for every $v\in V$. If a fractional matching assumes only 0, 1 values, then it is the characteristic function of a *matching*, i.e., a set of disjoint edges. Viewed as a vector $\bar{x}$ on E, a nonnegative real function x on E is a fractional matching precisely when $A\bar{x}\leqslant\bar{1}$ ($\bar{1}$ is the constant vector 1 on V).

A nonnegative function α on V is called a *w-cover* if $\alpha(u)+\alpha(v)\geqslant w(e)$ for every edge $e=(u,v)\in E$. Viewed as a vector $\bar{\alpha}$ on V, the condition reads $\bar{\alpha}A\geqslant\bar{w}$. Given a function α on V, we write $\operatorname{supp}(\alpha)=\{v\in V\colon \alpha(v)>0\}$ and $E(\alpha)=\{e=(u,v)\in E\colon \alpha(u)+\alpha(v)=w(e)\}$. For a set of edges F we write $s(F)=\bigcup F$.

The duality theorem of linear programming tells us that

$$\max\{\bar{w}\cdot\bar{x}\colon \bar{x}\geqslant 0,\ A\bar{x}\leqslant\bar{1}\}=\min\{\bar{1}\cdot\bar{\alpha}\colon \bar{\alpha}\geqslant 0,\ \bar{\alpha}A\geqslant\bar{w}\},\tag{1.1}$$

i.e.,

$$\max\left\{\sum_{e\in E} w(e)x(e)\colon x \text{ is a fractional matching}\right\}$$
$$=\min\left\{\sum_{v\in V}\alpha(v)\colon \alpha \text{ is a } w\text{-cover}\right\}.\tag{1.2}$$

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Moreover, since w is integral, by the unimodularity of A, there exist integral valued x and α at which the maximum and minimum, respectively, are attained in (1.2).

The first aim of this paper is to generalize (1.2) to the infinite case. It has been realized (see e.g. [2, 3]) that the correct way to extend LP duality to the infinite case is via the *complementary slackness conditions* which, for (1.2), say that if the maximum and the minimum in (1.2) are attained at x and α, respectively, then:

(a) $x(e)>0$ implies $\alpha(u)+\alpha(v)=w(e)$, where $e=(u,v)$, and

(b) $\alpha(u)>0$ implies $\sum_{u\in e} x(e)=1$.

Since, as mentioned, x above can be taken to be 0, 1 (i.e., a characteristic function of a matching), the slackness conditions can be summarized in the following theorem.

Theorem 1.1. *For Γ and w as above there exist a matching F and an integral w-cover α such that:*
(a) $F\subseteq E(\alpha)$,
(b) $\mathrm{supp}(\alpha)\subseteq s(F)$.

A pair (F,α) satisfying (a) and (b) is called *orthogonal.*

Not every infinite weighted bipartite graph has an orthogonal pair (matching, w-cover). To see this, take the graph whose sides are $A=\{a\}$, $B=\{b_i\colon i=1,2,\dots\}$, and whose edges are $\{(a,b_i)\colon i=1,2,\dots\}$ and $w((a,b_i))=i$. However, the result is true if we restrict the number of possible values of $w(e)$ on the edges e incident with any fixed vertex.

Theorem 1.2. *Let $\Gamma=(V,E)$ be a (possibly infinite) bipartite graph with a nonnegative integral weight function w on E, such that*

$$\max\{w(e)\colon v\in e\}<\infty \quad \textit{for every } v\in V. \tag{1.3}$$

Then there exists an orthogonal pair (F,α) where F is a matching and α is an integral w-cover.

For a weight function satisfying (1.3) we write $p(v)=\max\{w(e)\colon v\in e\}$ for every vertex v.

Theorem 1.2 is a generalization of the main theorem of [1], in which w is the constant function 1. In this case the theorem can be stated as follows.

Theorem 1.3 (Aharoni [1]). *In any bipartite graph there exist a matching F and a cover C such that C consists of the choice of precisely one vertex from each edge in F.*

(A *cover* is a set of vertices meeting all edges in the graph.)

If the sides of Γ are A and B, we write $\Gamma=(A,B,E)$. Given a subset X of B, we write $D(X)$ (or $D_\Gamma(X)$) for the set of all vertices in A joined only to vertices in X. A subset W of V is called *matchable* if there exists a matching F such that $W\subseteq s(F)$. The graph Γ is said to be *espousable* if A is matchable. Given a set H of edges and a subset U of V, we write $H\lceil U=\{e\in H\colon e\cap U\neq\emptyset\}$, $H[U]=\{v\in V\colon (v,u)\in H$ for some $u\in U\}$. For a matching F we have $s(F)=F[V]$. Theorem 1.3 easily implies the following corollary.

Corollary 1.3a. *If Γ is inespousable, then there exists a subset X of B such that $D(X)$ is unmatchable, and X is matchable into $D(X)$ (i.e., there exists a matching F such that $s(F)\cap B=X$ and $s(F)\cap A\subseteq D(X)$).*

Proof. Let F and C be as in the theorem, and let $X=B\cap C$. Since C is a cover, $D(X)=A\setminus C$, and thus $F\lceil X$ matches X into $D(X)$. If $D(X)$ was matchable by a matching, say, H, then $(H\lceil D(X))\cup(F\lceil (A\cap C))$ would be a matching of A, contrary to the assumption that Γ is inespousable. □

Remark. Theorem 1.3 can also be deduced from Corollary 1.3a, which was, in fact, the direction taken in [1].

A *w-packing* in the weighted graph Γ is a nonnegative integral function β on V such that $\beta(u)+\beta(v)\leqslant w(e)$ for every edge $e=(u,v)\in E$. A set H of edges is called a *vertex-cover* if $s(H)=V$. A vertex-cover H and a w-packing β are called *orthogonal* if: (a) $H\subseteq E(\beta)$ and (b) if $\beta(v)>0$, then v belongs to precisely one edge from H.

The finite case of the following theorem again follows directly from the duality theorem of linear programming.

Theorem 1.4. *In every (possibly infinite) bipartite graph with a nonnegative integral weight function w on its edge set, there exists an orthogonal pair (H,β) of a vertex-cover H and a w-packing β.*

The following lemma is a strengthening of the theorem of Cantor–Bernstein and is proved in the same way.

Lemma 1.5. *Let $\Gamma=(A,B,E)$ be a bipartite graph, and let $A'\subseteq A$, $B'\subseteq B$. If A' and B' are matchable, then so is $A'\cup B'$.*

2. Proof of Theorem 1.2

We may clearly assume that $w(e)>0$ for every $e\in E$. Let α_0 be defined by: $\alpha_0(b)=0$ for every $b\in B$, $\alpha_0(a)=p(a)$ for every $a\in A$. Clearly, α_0 is a w-cover. Let $E_0=E(\alpha_0)$ and let $\Gamma_0=(A,B,E_0)$.

Assume first that Γ_0 is espousable, and let J_0 be a matching of A in Γ_0. Then (J_0, α_0) is an orthogonal pair in Γ. Hence we may assume that Γ_0 is inespousable. By Corollary 1.3a there exist a subset X_0 of B such that $D_{\Gamma_0}(X_0)$ is unmatchable in Γ_0 and a matching $H_0 \subseteq E_0$ of X_0 into $D_{\Gamma_0}(X_0)$.

Let α_1 be defined by: $\alpha_1(b)=\alpha_0(b)+1$ if $b \in X_0$, $\alpha_1(a)=\alpha_0(a)-1$ if $a \in D_{\Gamma_0}(X_0)$, $\alpha_1(v)=\alpha_0(v)$ for $v \in V \setminus (X_0 \cup D_{\Gamma_0}(X_0))$. Let $A_1=\{a \in A\colon \alpha_1(a)>0\}$, $E_1=E(\alpha_1) \lceil A_1$ and let $\Gamma_1=(A_1, B, E_1)$. By the choice of X_0, α_1 is a w-cover in Γ.

Assume that Γ_1 is espousable. Note that all edges in E_0 adjacent to $D_{\Gamma_0}(X_0)$ are in $E(\alpha_1)$. Hence $H_0 \subseteq E(\alpha_1)$. By Lemma 1.5 it follows that there exists a matching $J_1 \subseteq E(\alpha_1)$ covering $X_0 \cup A_1$. Now, since $\mathrm{supp}(\alpha_1)=X_0 \cup A_1$, the pair (J_1, α_1) is an orthogonal pair as required in the theorem. Thus we may assume that Γ_1 is inespousable. By Corollary 1.3a, there exist therefore a subset X_1 of B such that $D_{\Gamma_1}(X_1)$ is unmatchable in Γ_1 and a matching $H_1 \subseteq E_1$ of X_1 into $D_{\Gamma_1}(X_1)$.

We continue in this way until, at a certain ordinal stage μ, the set A shrinks to a matchable set A_μ in $E(\alpha_\mu)$, which must happen because, at worst, eventually we must have $A_\mu=\emptyset$. Formally this is done as follows.

We define inductively w-covers α_ρ, subsets X_ρ of B and matchings J_ρ, where ρ is an ordinal which, as will be seen from the construction, cannot exceed $\max(\kappa^+, \aleph_0)$, where $\kappa=|V|$. Assume that α_ν has been defined for all $\nu<\rho$ and that $\alpha_\nu(v) \leqslant p(v)$ for every $v \in V$. If ρ is a limit ordinal, let $\alpha_\rho(a)=\inf\{\alpha_\nu(a)\colon \nu<\rho\}$ for every $a \in A$ and $\alpha_\rho(b)=\sup\{\alpha_\nu(b)\colon \nu<\rho\}$ for every $b \in B$.

Assume next that $\rho=\zeta+1$ is a successor ordinal, and that $X_\zeta, A_\zeta, E_\zeta$ and Γ_ζ have already been defined. Define then: $\alpha_\rho(v)=\alpha_\zeta(v)$ for all $v \in V \setminus (D_{\Gamma_\zeta}(X_\zeta) \cup X_\zeta)$; $\alpha_\rho(v)=\alpha_\zeta(v)+1$ if $v \in X_\zeta$; $\alpha_\rho(v)=\alpha_\zeta(v)-1$ if $v \in D_{\Gamma_\zeta}(x_\zeta)$.

For each ρ define: $A_\rho=\mathrm{supp}(\alpha_\rho) \cap A$, $E_\rho=E(\alpha_\rho) \lceil A_\rho$ and $\Gamma_\rho=(A_\rho, B, E_\rho)$. If Γ_ρ is espousable, terminate the process of definition. If not, then by Corollary 1.3a there exist a subset X_ρ of B such that $D_{\Gamma_\rho}(X_\rho)$ is unmatchable in Γ_ρ and a matching $H_\rho \subseteq E_\rho$ of X_ρ into $D_{\Gamma_\rho}(X_\rho)$.

Note that at each stage $D_{\Gamma_\rho}(X_\rho)$, being unmatchable, is nonempty, and hence at the $(\rho+1)$-stage α_ρ will decrease on some point of A. Hence the process of definition cannot go on for more than $|A|$ steps (assuming A to be infinite). Thus, for a certain ordinal $\mu<|A|^+$, the graph Γ_μ must be espousable, possibly for the reason that $A_\mu=\emptyset$. Let J be an espousal of Γ_μ.

Assertion 2.1. *$\alpha_\rho(b) \leqslant p(b)$ for every ordinal ρ and every $b \in B$.*

Proof. Suppose that the assertion fails for some $b \in B$, and let ρ be the first ordinal for which $\alpha_\rho(b)>p(b)$. Then, by the definition of α_ρ, ρ is a successor ordinal, i.e., $\rho=\theta+1$, where $\alpha_\theta(b)=p(b)$ and $b \in X_\theta$. If some edge $e=(a,b)$ belongs to $E(\alpha_\theta)$, then we must have $\alpha_\theta(a)=0$, i.e., $a \notin A_\theta$, implying $e \notin E_\theta$. Thus b is isolated in Γ_θ which precludes $b \in X_\theta$, since X_θ is matchable in Γ_θ. This contradiction proves the assertion. □

Assertion 2.1 proves the boundedness of $\alpha_\rho(b)$, which is necessary for the inductive definition of α_ρ. It also shows that for each $b\in B$ the number of ordinals ρ for which $b\in X_\rho$ cannot exceed $p(b)$. Hence, for each $b\in B\cap\text{supp}(\alpha)$, there exists an ordinal $\theta(b)$ for which $b\notin X_\theta$ for $\theta\geqslant\theta(b)$.

Let $\alpha=\alpha_\mu$, $\tilde{A}=A_\mu=A\cap\text{supp}(\alpha)$, $\tilde{B}=B\cap\text{supp}(\alpha)$.

Assertion 2.2. *If* $b,c\in\tilde{B}$ *and* $\theta(b)\neq\theta(c)$, then $H_{\theta(b)}(b)\neq H_{\theta(c)}(c)$.

Proof. Write $\theta(b)=\gamma$, $\theta(c)=\delta$. Assume, say, that $\gamma<\delta$. Let $H_\delta(c)=a$. Suppose also that $H_\gamma(b)=a$. Then $(a,b)\in E_\gamma$ and, since by the definition of γ we have $\alpha_\delta(b)=\alpha_\delta(b)$ and since α_δ is a w-cover, $(a,b)\in E_\delta$. However, the assumption that $\gamma<\delta$ implies that $b\notin X_\delta$ and therefore $a\notin D_{\Gamma_\delta}(X_\delta)$. This contradicts the assumption that $a=H_\delta(c)\in D_{\Gamma_\delta}(X_\delta)$. □

Define $H=\{H_{\theta(b)}\restriction\{b\}\colon b\in\tilde{B}\}$.

Assertion 2.3. *H is a matching.*

Proof. If $b\neq c\in\tilde{B}$ and $\theta(b)\neq\theta(c)$, then $H(b)\neq H(c)$ by Assertion 2.2. If $\theta(b)=\theta(c)$, then $H(b)\neq H(c)$ since $H_{\theta(b)}$ is a matching. □

We can now complete the proof of the theorem.

By the definition of the ordinals $\theta(b)$ there obtains $H\subseteq E_\mu$. Applying Lemma 1.5 to the graph Γ_μ we see that there exists a matching F in Γ_μ such that $\tilde{A}\cup\tilde{B}\subseteq s(F)$. The pair (F,α) is then orthogonal, as desired in the theorem.

3. Proof of Theorem 1.4

For a vertex $v\in V$ define $q(v)=\min\{w(e)\colon v\in e\}$. Let β_0 be the w-packing defined by $\beta_0(a)=0$ for all $a\in A$, $\beta_0(b)=q(b)$ for all $b\in B$. Let $Z_0=\{b\in B\colon \beta_0(b)=0\}$, $K_0=E(\beta_0)[Z_0]$, $A_0=A\setminus K_0$, $B_0=B\setminus Z_0$. Also let $E_0=E(\beta_0)\cap(A_0\times B_0)$ and $\Gamma_0=(A_0,B_0,E_0)$. Suppose that Γ_0 is espousable, and let F_0 be a matching of A in Γ_0. For each vertex $b\in B\setminus F_0[A]$ choose an edge $e_b\in E(\beta_0)$ containing b. Let $H_0=F_0\cup\{e_b\colon b\in B\setminus F_0[A]\}\cup(E(\beta_0)\restriction Z_0)$. Then (H_0,β_0) is an orthogonal pair (vertex-cover, w-packing).

We may thus assume that Γ_0 is inespousable. By Corollary 1.3a there exists then a subset X_0 of B_0 such that $D_{\Gamma_0}(X_0)$ is unmatchable and there exists a matching J_0 of X_0 into $D_{\Gamma_0}(X_0)$ in Γ_0. Let β_1 be the w-packing on Γ defined by $\beta_1(b)=\beta_0(b)-1$ for all $b\in X_0$ (note that $\beta_0(b)>0$); $\beta_1(a)=\beta_0(a)+1$ for all $a\in D_{\Gamma_0}(X_0)$; $\beta_1(v)=\beta_0(v)$ for all $v\in V\setminus(X_0\cup D_{\Gamma_0}(X_0))$.

Let $Z_1=\{b\in B\colon \beta_1(b)=0\}$, $K_1=E(\beta_1)[Z_1]$, $A_1=A_0\setminus K_1$, $B_1=B_0\setminus Z_1$, $E_1=E(\beta_1)\cap(A_1\times B_1)$ and $\Gamma_1=(A_1,B_1,E_1)$. Assume that Γ_1 is espousable. Since $J_0\subseteq E(\beta_1)$, this means that both A_1 and X_0 are matchable in the graph $(A,B,E(\beta_1))$. By Lemma 1.5 it follows that there exists a matching F_1 of $A_1\cup X_0$ in Γ contained in

$E(\beta_1)$. For each vertex $b \in B \setminus s(F_1)$ there obtains $\beta_1(b) = q(b)$ (since $b \notin X_0$). Hence there exists an edge $e = e(b) = (a, b)$ such that $w(e) = \beta_1(b) = q(b)$. Since $b \notin X_0$, we have $a \notin D_{\Gamma_0}(X_0)$, hence $\beta_1(a) = 0$. Let $F'_1 = F_1 \cup \{e(b): b \in B \setminus s(F_1)\}$. By the above, no vertex v with $\beta_1(v) > 0$ is covered by F'_1 more than once.

If $a \in A \setminus s(F'_1)$, then $a \in K_1$. Hence, for each $a \in A \setminus s(F'_1)$, there exists a vertex $z \in Z_1$ such that $(a, z) \in E(\beta_1)$. Let $f(a) = (a, z)$. Let $H_1 = F'_1 \cup \{f(a): a \in A \setminus s(F'_1)\}$. Then H_1 is a vertex-cover and, by the above, (H_1, β_1) is an orthogonal pair, as desired.

We may thus assume that Γ_1 is inespousable. By Corollary 1.3a there exists then $X_1 \subseteq B_1$ such that $D_{\Gamma_1}(X_1)$ is unmatchable in Γ_1 and X_1 is matchable in Γ_1 into $D_{\Gamma_1}(X_1)$.

In general, we construct, by induction on ρ, the following sequences:

β_ρ — a w-packing,
A_ρ — a subset of A,
B_ρ — a subset of B,
X_ρ — a subset of B_ρ,
J_ρ — a matching in Γ_ρ of X_ρ into $D_{\Gamma_\rho}(X_\rho)$,
where $\Gamma_\rho = (A_\rho, B_\rho, E(\beta_\rho) \cap (A_\rho \times B_\rho))$.

The definition is as follows: for $\rho = 0, 1$ all objects have already been defined. Let $\rho > 1$ and assume that these objects have been defined for all $\zeta < \rho$, and furthermore that for each $a \in A$ the sequence $\{\beta_\zeta(a): \zeta < \rho\}$ is nondecreasing and for each $b \in B$ the sequence $\{\beta_\zeta(b): \zeta < \rho\}$ is nonincreasing. If ρ is a limit ordinal, define for $a \in A$, $b \in B$: $\beta_\rho(a) = \sup(\beta_\zeta(a): \zeta < \rho\}$ (note that since β_ζ is a w-packing, $\beta_\zeta(a) \leqslant q(a)$ for all $\zeta < \rho$); $\beta_\rho(b) = \inf\{\beta_\zeta(b): \zeta < \rho\}$; $A_\rho = \bigcap\{A_\zeta: \zeta < \rho\}$ and $B_\rho = \bigcap\{\beta_\zeta: \zeta < \rho\}$.

If $\rho = \zeta + 1$ is a successor ordinal, let $\beta_\rho(a) = \beta_\zeta(a) + 1$ for all $a \in D_{\Gamma_\zeta}(X_\zeta)$; $\beta_\rho(b) = \beta_\zeta(b) - 1$ for all $b \in X_\zeta$; $\beta_\rho(v) = \beta_\zeta(v)$ for all other vertices v of V.

Also, in this case, let $Z_\rho = \{b \in B_\zeta: \beta_\rho(b) = 0\}$, $B_\rho = B_\zeta \setminus Z_\rho$, $A_\rho = A_\zeta \setminus E(\beta_\rho)[Z_\rho]$, $E_\rho = E(\beta_\rho) \cap (A_\rho \times B_\rho)$ and $\Gamma_\rho = (A_\rho, B_\rho, E_\rho)$.

If Γ_ρ is espousable, terminate the process of definition (and then X_ρ, J_ρ are undefined). If Γ_ρ is inespousable, then by Corollary 1.3a there exists a subset X_ρ of B_ρ such that $D_{\Gamma_\rho}(X_\rho)$ is unmatchable in Γ_ρ, and there exists a matching J_ρ of X_ρ into $D_{\Gamma_\rho}(X_\rho)$ in Γ_ρ.

Assertion 3.1. *β_ρ is a w-packing.*

Proof. By induction on ρ. For ρ a limit ordinal, the assertion follows directly from the induction hypothesis. Let $\rho = \zeta + 1$ and let $a \in A$, $b \in B$. By the definition of β_ρ we have $\beta_\rho(a) \geqslant \beta_\zeta(a)$ and hence, by the induction hypothesis, $\beta_\rho(a) \geqslant 0$.

If $b \notin X_\zeta$, then $\beta_\rho(b) = \beta_\zeta(b)$; hence $\beta_\rho(b) \geqslant 0$ by the induction hypothesis. If $b \in X_\zeta$, then $b \in B_\zeta$, implying $\beta_\zeta(b) > 0$; hence again $\beta_\rho(b) \geqslant 0$. Thus $\beta_\rho(v) \geqslant 0$ for all $v \in V$.

Assume now that $e = (a, b) \in E$. If $\beta_\rho(a) = \beta_\zeta(a)$, then, since $\beta_\rho(b) \leqslant \beta_\zeta(b)$, we have $\beta_\rho(a) + \beta_\rho(b) \leqslant \beta_\zeta(a) + \beta_\zeta(b) \leqslant w(e)$ by induction hypothesis. If $\beta_\rho(a) = \beta_\zeta(a) + 1$, then $a \in D_{\Gamma_\zeta}(X_\zeta)$, and hence either $(a, b) \notin E_\zeta$ or $b \in X_\zeta$. In the first case $\beta_\zeta(a) + \beta_\zeta(b) < w(e)$, implying $\beta_\rho(a) + \beta_\rho(b) \leqslant w(e)$. In the second case $\beta_\rho(b) = \beta_\zeta(b) - 1$, and hence $\beta_\rho(a) + \beta_\rho(b) = \beta_\zeta(a) + \beta_\zeta(b) \leqslant w(e)$, again by the assumption that β_ζ is a w-packing. This proves the assertion. $\square$

In every step ρ in which Γ_ρ is inespousable, $D_{\Gamma_\rho}(X_\rho)$ is unmatchable and therefore nonempty. Hence $\beta_{\rho+1}(a)>\beta_\rho(a)$ for at least one $a\in A$. By Assertion 3.1 $\beta_\rho(a)\leqslant q(a)$ for all $a\in A$. Hence, if A is infinite, Γ_ρ cannot be inespousable for more than $|A|$ many values of ρ. Thus, for some ordinal $\theta<|A|^+$, the graph Γ_θ is espousable.

Let $X=\bigcup\{X_\rho\colon \rho<\theta\}$. For each $x\in X$ let $\xi(x)=\max\{\rho\colon x\in X_\rho\}$. Note that this maximum exists since $\beta_\theta(x)\geqslant 0$. Let $J=\{(J_{\xi(x)}(x),x)\colon x\in X\}$.

Assertion 3.2. $J\subseteq E(\beta_\theta)$.

Proof. Let $x\in X$ and write $\xi=\xi(x)$, $J_\xi(x)=a$. By the definition of J_ξ, we have $(a,x)\in E(\beta_\xi)$. By the definition of $\xi(x)$, for all $\zeta>\xi$ there holds $x\notin X_\zeta$ and hence $\beta_\zeta(x)=\beta_{\xi+1}(x)=\beta_\xi(x)-1$. One now shows by induction on ζ $(\xi\leqslant\zeta\leqslant\theta)$ that $(a,x)\in E(\beta_\zeta)$. Assuming that this is true for $\xi<\zeta<\theta$, it follows that if $x\notin B_\zeta$, then $a\notin D_{\Gamma_\zeta}(X_\zeta)$, for otherwise we would have $\beta_{\zeta+1}(a)=\beta_\zeta(a)+1$ and hence $\beta_{\zeta+1}(a)+\beta_{\zeta+1}(x)>w((a,x))$, contradicting Assertion 3.1. Thus $\beta_{\zeta+1}(a)=\beta_\zeta(a)$. Assume that $x\in B_\zeta$. If $a\in A_\zeta$, then again $a\notin D_{\Gamma_\zeta}(X_\zeta)$ (since $x\notin X_\zeta$). Thus $\beta_{\zeta+1}(a)=\beta_\zeta(a)$. If $a\notin A_\zeta$, then again $\beta_{\zeta+1}(a)=\beta_\zeta(a)$ and in both cases $(a,x)\in E(\beta_{\zeta+1})$. For ζ a limit ordinal, the claim follows from the induction hypothesis. Putting $\zeta=\theta$ proves the assertion. □

Assertion 3.3. *J is a matching.*

Proof. Similar to the proof of Assertion 2.3. □

By Assertions 3.2 and 3.3, the assumption that Γ_θ is espousable and Lemma 1.5, there exists a matching $F_\theta\subseteq E(\beta_\theta)$ of $A_\theta\cup X$. For each $b\in B\setminus s(F_\theta)$ there holds $\beta_\theta(b)=q(b)$, since $b\notin X$. Choose $a\in A$ such that $e(b)=(a,b)$ satisfies $w(e)=q(b)$. Since β_θ is a w-packing, $\beta_\theta(a)=0$. Let $F'_\theta=F_\theta\cup\{e(b)\colon b\in B\setminus s(F_\theta)\}$. Let $a\in A\setminus s(F'_\theta)$. Since $a\notin A_\theta$, there holds $a\in E(\beta_\zeta)[Z_\zeta]$ for some $\zeta<\theta$. Choose a vertex $b\in Z_\zeta$ such that $(a,b)\in E(\beta_\zeta)$ and let $g(a)=(a,b)$. By the definition of β_ρ $(\rho>\zeta)$ we have $\beta_\rho(a)=\beta_\zeta(a)$, $\beta_\rho(b)=\beta_\zeta(b)$, and hence $g(a)\in E(\beta_\rho)$. In particular, $g(a)\in E(\beta_\theta)$.

Let $H=F'_\theta\cup\{g(a)\colon a\in A\setminus s(F'_\theta)\}$. Clearly H is a vertex-cover.

By the above, $H\subseteq E(\beta_\theta)$. The addition of the edges $e(b)$ to F_θ yields vertices a with degree larger than one, but this happens only for a satisfying $\beta_\theta(a)=0$. Again, the addition of the edges $g(a)$ to F'_θ generates degree larger than one only at vertices b satisfying $\beta_\theta(b)=0$. Thus (H,β_θ) is an orthogonal pair, as required in the theorem.

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